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ON THE ROTATION OF THE UNIVERSE

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Summary

This paper discusses the possibility that the Universe could possess a large scale, homogeneous, vorticity. The limit that can be placed on the present value of this vorticity from the present observations of the microwave background is between 10^{-14} and 7×10^{-17} rad yr^{-1} if the Universe is spatially closed and about 2×10^{-46}/(present density in g cm^{-3}) rad yr^{-1} if it is open. If the observations were extended over the whole sky it might be possible to decide whether, in fact, the Universe is open. It is also shown that viscosity tends to reduce vorticity though not sufficiently to produce the present limits from arbitrary initial conditions unless the present density has the very low value of $2 \cdot 5 \times 10^{-33}$ g cm^{-3}.

1. *Introduction.* Recent observations by Partridge & Wilkinson (1967) indicate that the microwave background radiation is isotropic to within 0·5 per cent round a circle near the equator. Whatever explanation one accepts of the origin of this radiation, it seems to be fairly generally agreed that it must have propagated freely towards us from a distance of the order of the Hubble radius. If one now assumed that the radiation would appear equally isotropic to other observers round other circles in the sky, it follows that, on a large scale, the Universe must be described by one of the Robertson–Walker models (Hawking & Ellis (1968)). Departures from such a model may be divided roughly into inhomogeneities and large scale anisotropy. The effect of the former on the background radiation have been investigated by Sachs & Wolfe (1967), Wolfe (1968) and Rees & Sciama (1968) while that of the latter has been discussed by Thorne (1967) and Misner (1968) who considered certain spatially homogeneous models in which the matter has shear but not vorticity. One assumes spatial homogeneity because some such simplification is necessary to make the problem mathematically tractable and because there is some philosophical justification for it from the Copernian (or modesty) Principle that we do not occupy a special position in the Universe. However, one does not know that the vorticity is necessarily zero and so, in this paper, I shall consider more general spatially homogeneous models in which there is both shear and vorticity and shall discuss what limits can be placed on the values of these from the observations.

By vorticity one means the rotation of ' nearby ' matter about an observer moving with the matter, relative to an inertial frame defined by gyroscopes. (Here ' nearby ' should be interpreted as meaning at distances of about a hundred megaparsecs, to be near compared to the Hubble radius but far compared to the length scales ot local phenomena such as the rotation of the Galaxy.) Thus in a sense, the whole Universe would be rotating, though, as the model is homogeneous, there would be no centre of rotation. Such rotation, if it existed, would be of great interest for the dynamical effects it could have on the Universe and on the formation of galaxies and for its relation to Mach's Principle. This states that the local inertial frame should be non-rotating with respect to distant matter.

Quite how this comparison is supposed to be made is not clear. Nevertheless it would seem that this Principle is incompatible with the existence of a homogeneous non-zero vorticity. Unfortunately it is difficult to detect vorticity directly unless it is comparatively large as the only way of measuring the transverse velocities of objects at distances of a hundred megaparsecs is by the transverse Doppler effect. As this depends on v^2/c^2 it is not very sensitive and the best limit that can be placed on the vorticity by direct observation of galaxies is that it is less than 7×10^{-11} rad yr^{-1} (Kristian & Sachs 1966) which corresponds roughly to a transverse velocity of the order of c at a distance of the Hubble radius. It should be emphasized that this is a very high limit and that if the vorticity had this value, large scale centrifugal effects would be comparable at the present time to gravitational forces. However, it will be shown in this paper that the observed isotropy of the background radiation enables us to place a much lower limit on the possible present day value of the vorticity. Roughly speaking, the existence of vorticity implies that, relative to the space-like surfaces of homogeneity, the matter must have a peculiar velocity of magnitude equal to the vorticity times some characteristic radius of curvature of the surface of homogeneity. One might expect this peculiar velocity to produce a 24 h (dipole) component of anisotropy in the microwave background. Of course there will be contributions to the 24 h component from the motion of the Earth round the Sun, the Sun round the Galaxy, the Galaxy round the super-cluster etc. (Sciama (1967); Stewart & Sciama (1967)). Thus one could not be sure that the observed 24 h component was ascribable to vorticity but the fact that it is less than 0·1 per cent (Partridge & Wilkinson (1967), indicates that the peculiar velocity resulting from vorticity is probably less than 300 km s^{-1}. As the characteristic radius of curvature will be of the order of the Hubble radius or bigger, it follows that the vorticity is less than about 7×10^{-14} rad yr^{-1}. Of course one must also take into account the peculiar velocity of the matter where the radiation was emitted or last scattered. In a nearly isotropic universe this velocity will vary inversely as the average length scale. Thus if the radiation had been last scattered when the density was much higher than it is now (and this is the case in most suggested theories), the peculiar velocity would have been much higher and would have given rise to a Doppler shift in the radiation which varied with the angle of observation. The relation between this red shift and the angle depends on the particular type of homogeneous solution. In a high density, spatially closed, model this relation is more or less what one would expect in flat space and enables one to put a limit on the vorticity of about 7×10^{-14} $(\lambda-1)^{-1}$ rad yr^{-1} where λ is the ratio of the present radius of the Universe to that when the radiation was last scattered. However, in a low density, open, model the relation is more complicated and present observations only place a limit on the vorticity of about 2×10^{-15} rad yr^{-1}. The observed temperature of the radiation would have anisotropic components of the form, cot $(\theta/2)$ cos ϕ and cot^2 $(\theta/2)$ cos 2ϕ for θ greater than 5 minutes of arc, where θ is the angle between the direction of observation and a preferred direction. The first of these components would be related to the vorticity and the second to the shear. If such a pattern of variation were detected by making observations over the whole sky, it would be a definite indication that we live in an open universe (cf. Novikov (1968)).

Misner has proposed a ' chaotic cosmology ' programme the aim of which is to show that there are dissipation mechanisms which damp out practically all initial irregularities and anisotropies and cause any model which continues expanding to evolve to resemble the observed universe. As a first step in this programme Misner (1968) has shown that viscosity in the early Universe could reduce the shear from a very large value to the present very small one. In this paper it will be shown that viscosity also reduces homogeneous vorticity though it seems that it cannot do so sufficiently to account for the present observations starting from arbitrary initial conditions, unless the present density is less than the very low value of $2\cdot 5\times 10^{-33}$ g cm^{-3}.

The plan of the paper is as follows. Section 2 outlines the approach to spatially homogeneous models of Heckmann & Schucking (1962). The equations of geodesics and of world-lines are derived in Section 3 and an approximate method is given to calculate what the observed temperature of the radiation should be. In Section 4 the field equations are analysed using techniques similar to those of Misner (1968). In Sections 5 and 6 models of Bianchi types IX and V are considered in detail. These generalize the closed and open Robertson–Walker models respectively. The effects of viscosity on the vorticity is considered in Section VII.

2. *Spatially homogeneous models.* The models that will be considered in this paper admit a three parameter group of isometries simply transitive on a family of spacelike hypersurfaces. These surfaces will be labelled by a parameter t chosen so that $g^{\mu\nu}t_{;\mu}t_{;\nu} = -1$ where a semicolon indicates the covariant derivative.

The metric of such a model has the form

$$g_{\mu\nu} = -n_\mu n_\nu + g_{AB}E^A{}_\mu E^B{}_\nu, \tag{2.1}$$

where $n_\mu = -t_{;\mu}$ is the normal to the surfaces of homogeneity, g_{AB} is a 3×3 matrix which depends only on t and $E^A{}_\mu$ are three invariant covector fields in the surfaces of homogeneity. (Greek indices run from 0 to 3, Latin indices from 1 to 3.) They obey the relations

$$E^A{}_{\mu;\nu} - E^A{}_{\nu;\mu} = C_{BC}{}^A E^B{}_\mu E^C{}_\nu, \tag{2.2}$$

where $C_{BC}{}^A$ are the structure constants of the group of isometries. They will be of one of nine possible types first classified by Bianchi (see Taub (1951) for the canonical forms and Ellis & MacCallum (1968) for further details of classification). The nine types can be divided into class A or class B according to whether $C_{AC}{}^A$ vanishes or not.

Since the Universe is so nearly isotropic, this paper will be mainly concerned with those groups that permit (but do not require) the metric to be that of one of the Robertson–Walker models. For $k = +1$, the group will be of Bianchi type IX with structure constants,

$$C_{BC}{}^A = \epsilon_{ABC}, \tag{2.3}$$

where ϵ_{ABC} is the permutation symbol. This group belongs to class A. For $k = -1$, the group can be of Bianchi type V or VII (Ellis & MacCallum (1968)) both of which belong to class B. It is this difference in class which accounts for the different nature of the predicted observations in an open universe. Type VII is similar to V but has an extra twist in the invariant covectors. Since the magnitude of this twist is an arbitrary parameter and since type V is simpler only it will be considered in detail in this paper. The non-zero structure constants for type V are:

$$C_{2\,1}{}^{2} = -C_{1\,2}{}^{2} = C_{3\,1}{}^{3} = -C_{1\,3}{}^{3} = 1. \tag{2.4}$$

For $k = 0$, the group will be of Bianchi type I (all structure constants zero).* Such models have been considered by Thorne (1967) and Misner (1968) but are not of interest here since they do not allow the matter to have vorticity.

One may define three invariant vector fields $E_A{}^\mu$ in the surfaces of homogeneity which are dual to the $E^A{}_\mu$:

$$E^A{}_\mu E_B{}^\mu = \delta_B{}^A. \tag{2.5}$$

Then

$$E^A{}_\mu E_A{}^\nu = \delta_\mu{}^\nu + n^\nu n_\mu. \tag{2.6}$$

Any tensor field can be expressed in terms of its components with respect to the $E^A{}_\mu$, the $E_A{}^\mu$, n_μ and n^μ. If the field is invariant under the group of isometries, the components will be functions of t only. For example, the flow-vector of the matter can be expressed as

$$U^\mu = U^0 n^\mu + U^A E_A{}^\mu,$$

where

$$U^0 = -U^\mu n_\mu \text{ and } U^A = U^\nu E^A{}_\nu$$

depend only on t. Capital Latin indices may be lowered and raised by the matrix $g_{AB} = g_{\mu\nu} E_A{}^\mu E_B{}^\nu$ and its inverse $g^{AB} = g^{\mu\nu} E^A{}_\mu E^B{}_\nu$. Following Misner (1968), the matrix g_{AB} will be split into its volume and distortion parts:

$$g_{AB} = e^{2\alpha}\,(e^{2\beta})_{AB}, \tag{2.7}$$

where β is a symmetric, trace free 3×3 matrix and $e^{2\beta}$ is the series $\Sigma(r!)^{-1}(2\beta)^r$. Then an orthonormal basis $X^\gamma{}_\mu$ can be defined where $X^0{}_\mu = -n_\mu$ and

$$X^i{}_\mu = e^{-\alpha}(e^{-\beta})_{iA} E^A{}_\mu.$$

Components from 1 to 3 in this basis will be denoted by lower case Latin indices. They remain unchanged on raising or lowering while the 0 component changes sign. The Ricci rotation coefficients for this basis are defined by

$$X^\gamma{}_{\mu;\nu} = \Gamma_{\delta\epsilon}{}^\gamma X^\delta{}_\mu X^\epsilon{}_\nu, \tag{2.8}$$

and satisfy $\Gamma_{\delta\epsilon\gamma} + \Gamma_{\gamma\epsilon\delta} = 0$.

Therefore

$$\Gamma_{\gamma\delta\epsilon} = X_{\gamma[\mu;\nu]} X_\delta{}^\mu X_\epsilon{}^\nu + X_{\epsilon[\mu;\nu]} X_\gamma{}^\mu X_\delta{}^\nu - X_{\delta[\mu;\nu]} X_\epsilon{}^\mu X_\gamma{}^\nu, \tag{2.9}$$

where the convention has been adopted that square brackets around indices indicates antisymmetrization and round brackets, symmetrization. By equation (2.2),

$$\begin{aligned}
\Gamma_{ij0} &= -\Gamma_{0ij} = \alpha'\delta_{ij} + \sigma_{ij},\\
\Gamma_{i00} &= -\Gamma_{00i} = 0,\\
\Gamma_{i0j} &= -\tau_{ij},\\
\Gamma_{ijk} &= \tfrac{1}{2} e^{-\alpha} (e^{\beta})_{FA} (e^{-\beta})_{GB} (e^{-\beta})_{HC} C_{BC}{}^{A}\,[\delta_{Fi}\delta_{Gj}\delta_{Hk} + \delta_{Fk}\delta_{Gi}\delta_{Hj} - \delta_{Fj}\delta_{Gk}\delta_{Hi}],
\end{aligned} \tag{2.10}$$

* It would also be possible for the group to be type VII for a special value of the arbitrary parameter in the group but such a model would not seem a natural generalization of the $k = 0$ Robertson–Walker model.

where prime denotes differention by t, $\sigma_{ij} = (e^{\beta})'_{k(j}(e^{-\beta})_{i)k}$ represents the shear of the normals n_{μ} and $\tau_{ij} = (e^{\beta})'_{k[i}(e^{-\beta})_{j]k}$ represents the extent to which $(e^{\beta})'$ does not commute with e^{β}.

3. *Geodesics and world lines*

3.1 *Geodesics*. The parallelly transported tangent vector l^{μ} to a geodesic obeys $l^{\mu}{}_{;\nu} = 0$. In components with respect to the orthonormal basis this gives

$$l^0 l^{i\prime} + \Gamma^i{}_{jk} l^j l^k + \Gamma^i{}_{0j} l^0 l^j + \Gamma^i{}_{j0} l^j l^0 = 0, \qquad (3.1)$$

where $l^0 = [l^i l^i]^{1/2}$ for null geodesics and $[1 + l^i l^i]^{1/2}$ for timelike geodesics. This equation has a simple form in terms of components with respect to the $E^{\mathrm{A}}{}_{\mu}$:

$$l_{\mathrm{A}}' = (l^0)^{-1} C_{C\mathrm{A}}{}^{\mathrm{B}} l_{\mathrm{B}} l_{\mathrm{D}} e^{-2\alpha}(e^{-2\beta})_{\mathrm{CD}}. \qquad (3.2)$$

From this it can be seen that l_{A} is nearly constant for a time-like geodesic for which $[l^i l^i]^{1/2}$ is small, i.e. which has a low velocity relative to the surfaces of homogeneity. In type IX l_{A} is exactly constant for timelike or null geodesics if it lies along a principal direction of β. If it does not, it precesses like the angular velocity of a free body with unequal moments of inertia. Provided β is small the rate of precession is slow. In type V, l_{A} is constant only if it lies in the positive or negative 1 direction. If it does not, it turns towards the positive 1 direction. This happens whatever the value of β and is a result of the fact that the invariant vector field, $E_1{}^{\mu}$ is asymptotically radial about any point.

3.2 *Observations*. The background radiation may be considered in a first approximation as coming in from a surface of homogeneity in our past corresponding to the last time the radiation was scattered. The observed temperature in a given direction will be $T_R = T_{\mathrm{E}}(1+z)^{-1}$ where T_{E} is the temperature of the surface and z is its redshift in that direction. This is given by

$$1 + z = U_{\mathrm{E}}{}^{\mu} K_{\mu} / U_R{}^{\mu} K_{\mu},$$

where $U_R{}^{\mu}$ is the velocity vector of the observer, $U_{\mathrm{E}}{}^{\mu}$ is the velocity vector of the matter at the emitting surface and K^{μ} is the tangent vector to the null geodesic from the observer in the given direction. Thus

$$T_R = T_{\mathrm{E}}(U_R{}^0 + K_R{}^i U_R{}^i)((K_{\mathrm{E}}{}^j K_{\mathrm{E}}{}^j)^{1/2} U_{\mathrm{E}}{}^0 + K_{\mathrm{E}}{}^i U_{\mathrm{E}}{}^i)^{-1}, \qquad (3.3)$$

where $K_R{}^0$ has been taken to be minus one. The term $K_R{}^i U_R{}^i$ gives the dipole variation from the present peculiar velocity of the matter. The term involving $(K_{\mathrm{E}}{}^i K_{\mathrm{E}}{}^i)^{1/2}$ gives the red-shift resulting from the expansion of the Universe. From equation (3.2) it follows that

$$(K^i K^i e^{2\alpha})' = -2 K^j K^k \sigma_{jk} e^{2\alpha}. \qquad (3.4)$$

Thus if σ is small, $(K^i K^i)^{1/2}$ is approximately proportional to $e^{-\alpha}$ as in Robertson–Walker models. The first order correction to this $e^{-\alpha}$ law may be calculated from equation (3.4) using on the right the values that K^j would have in an isotropic universe. The term $K_{\mathrm{E}}{}^i U_{\mathrm{E}}{}^i$ gives the Doppler shift arising from the peculiar velocity of the emitting matter. To first order it will be given by replacing $K_{\mathrm{E}}{}^i$ by its value in an isotropic universe.

3.3 *Energy momentum tensor*. At the present epoch the dominant contribution to the energy-momentum tensor comes from non-relativistic matter, i.e.

$$T^{\mu\nu} = \mu_b U^\mu U^\nu,$$

where μ_b is the baryon density and U^μ is the flow-vector. From the conservation equations it follows that the flow-lines are geodesics and that

$$(\mu_b U^0 e^{3\alpha})' = -\mu_b e^\alpha C_{\mathrm{AB}}{}^{\mathrm{A}} U_{\mathrm{C}} (e^{-2\beta})_{\mathrm{BC}}. \tag{3.5}$$

Thus μ_b is proportional to $(U^0 e^{3\alpha})^{-1}$ in models of class A. T^{00} can be expressed as $a_b e^{-3\alpha}(1+V_b)$ where a_b is constant in class A and

$$V_b = U^0 - 1 = [U_{\mathrm{A}} U_{\mathrm{B}} e^{-2\alpha}(e^{-2\beta})_{\mathrm{AB}} + 1]^{1/2} - 1$$

is the kinetic energy associated with the peculiar velocity of the matter relative to the surfaces of homogeneity. The anisotropic part of the spatial components of the energy momentum tensor is

$$T_{ij} - \frac{1}{3} T_{kk}\delta_{ij} = -a_b e^{-3\alpha} \frac{\partial V_b}{\partial \beta_{ij}} \tag{3.6}$$

where α and U_{A} are held fixed in the partial derivative. This relation will be used in the next section to obtain a Lagrangian for the field equations.

In the earlier stages of the Universe the matter will be dominated by photons and neutrinos. Because of Compton scattering, the photons will maintain an almost isotropic distribution about the flow-vector of the matter. Their energy–momentum tensor will be

$$T^{\mu\nu} = \tfrac{4}{3}\mu_p U^\mu U^\nu + \tfrac{1}{3}\mu_p g^{\mu\nu},$$

where the energy density μ_p obeys

$$[\mu_p (U^0)^{4/3} e^{4\alpha}]' = -\tfrac{4}{3}\mu_p (U^0)^{1/3} C_{\mathrm{AB}}{}^{\mathrm{A}} U_{\mathrm{C}} e^{2\alpha} (e^{-2\beta})_{\mathrm{BC}}. \tag{3.7}$$

The radiation pressure will cause acceleration of the flow lines:

$$\tfrac{4}{3}\mu_p U^\mu{}_{;\nu} U^\nu = -\tfrac{1}{3}\mu_{p;\nu} h^{\mu\nu}, \qquad h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu.$$

By homogeneity, $\mu_{p;\nu} = -\mu_p' n_\nu$. Thus

$$[\mu_p^{1/4} U_{\mathrm{A}}]' = \mu_p^{1/4} (U^0)^{-1} C_{\mathrm{CA}}{}^{\mathrm{B}} U_{\mathrm{B}} U_{\mathrm{D}} e^{-2\alpha} (e^{-2\beta})_{\mathrm{CD}}. \tag{3.8}$$

This shows that for low velocities the magnitude of the velocity is almost independent of α. The anisotropic part of the spatial components of the energy–momentum tensor is

$$T_{ij} - \frac{1}{3} T_{kk}\delta_{ij} = -2 a_p e^{-4\alpha} \frac{\partial V_p}{\partial \beta_{ij}}, \tag{3.9}$$

where $a_p = \mu_p (U^0)^{4/3} e^{4\alpha}$ is a constant in class A models and

$$V_p = (U^0)^{2/3} - 1 = [U_{\mathrm{A}} U_{\mathrm{B}} e^{-2\alpha} (e^{-2\beta})_{\mathrm{AB}} + 1]^{1/3} - 1.$$

The effect of the neutrinos will not be considered in this paper except during the period around 10^{10} °K when they produce a viscosity by collisions with electrons and positrons. The damping or otherwise of vorticity by this viscosity will be investigated in Section VII.

The vorticity vector of the matter is defined as $\omega^{\mu} = \eta^{\mu\nu\lambda\rho}U_{\nu}U_{\lambda;\rho}$. Thus

$$\omega^{\mathrm{A}} = e^{-3\alpha}\epsilon_{\mathrm{ABC}}[\tfrac{1}{2}C_{\mathrm{BC}}{}^{\mathrm{D}}U_{\mathrm{D}}U^{0}+U_{\mathrm{B}}U_{\mathrm{C}}'], \quad (3.10)$$

$$\omega^{0} = \tfrac{1}{2}e^{-3\alpha}\epsilon_{\mathrm{ABC}}C_{\mathrm{BC}}{}^{\mathrm{D}}U_{\mathrm{A}}U_{\mathrm{D}}. \quad (3.11)$$

This gives the relation between the peculiar velocity and the vorticity. For low velocities, $\omega^{\mathrm{A}} = e^{-3\alpha}U_{\mathrm{A}}$ in type IX and $\omega^{1} = 0$, $\omega^{2} = e^{-3\alpha}U_{3}$, $\omega^{3} = -e^{-3\alpha}U_{2}$ in type V. Thus for small U, the magnitude of the vorticity goes as $e^{-2\alpha}$ when the Universe is matter dominated and as $e^{-\alpha}$ when it is radiation dominated. This latter rate of growth with increasing density is so slow that if vorticity were not dynamically important back to the time when the Universe was radiation dominated, it would probably not have been important at any earlier time.

4. *The field equations.* In the orthonormal basis the components of the Ricci tensor are

$$R_{0}{}^{0} = 3\alpha''+3(\alpha')^{2}+\sigma_{ij}\sigma_{ij}, \quad (4.1)$$

$$R_{i}{}^{0} = e^{-\alpha}[(e^{-\beta}\sigma e^{\beta})_{\mathrm{BA}}C_{\mathrm{BC}}{}^{\mathrm{A}}(e^{-\beta})_{\mathrm{C}i}-\sigma_{ij}(e^{-\beta})_{j\mathrm{C}}C_{\mathrm{AC}}{}^{\mathrm{A}}], \quad (4.2)$$

$$R_{ij} = R_{ij}^{*}+[\alpha''+3(\alpha')^{2}]\delta_{ij}+(\sigma_{ij}'+3\alpha'\sigma_{ij}+\sigma_{ik}\tau_{kj}-\tau_{ik}\sigma_{kj}), \quad (4.3)$$

where R_{ij}^{*} is the Ricci tensor of the surfaces of homogeneity;

$$\begin{aligned} R_{ij}^{*} = -\tfrac{1}{4}e^{-2\alpha}\{&2C_{\mathrm{BC}}{}^{\mathrm{A}}C_{\mathrm{DA}}{}^{\mathrm{C}}(e^{-\beta})_{\mathrm{B}i}(e^{-\beta})_{\mathrm{D}j} \\ &+C_{\mathrm{BC}}{}^{\mathrm{A}}C_{\mathrm{EF}}{}^{\mathrm{D}}(e^{-2\beta})_{\mathrm{BE}}[2(e^{2\beta})_{\mathrm{DA}}(e^{-\beta})_{\mathrm{C}i}(e^{-\beta})_{\mathrm{F}j}-(e^{-2\beta})_{\mathrm{CF}}(e^{\beta})_{\mathrm{A}i}(e^{\beta})_{\mathrm{D}j}] \\ &+2C_{\mathrm{AB}}{}^{\mathrm{A}}C_{\mathrm{DE}}{}^{\mathrm{C}}(e^{-2\beta})_{\mathrm{BE}}[(e^{\beta})_{\mathrm{C}i}(e^{-\beta})_{\mathrm{D}j}+(e^{\beta})_{\mathrm{C}j}(e^{-\beta})_{\mathrm{D}i}]\}. \end{aligned} \quad (4.4)$$

The curvature scalar is

$$R = 6\alpha''+12(\alpha')^{2}+\sigma_{ij}\sigma_{ij}+R^{*}. \quad (4.5)$$

Thus the field equations are*

$$3(\alpha')^{2}-\tfrac{1}{2}\sigma_{ij}\sigma_{ij}+\tfrac{1}{2}R^{*} = 8\pi T^{00} \quad (4.6)$$

$$e^{-\alpha}[(e^{-\beta}\sigma e^{\beta})_{\mathrm{BA}}C_{\mathrm{BC}}{}^{\mathrm{A}}(e^{-\beta})_{\mathrm{C}i}-\sigma_{ij}(e^{-\beta})_{j\mathrm{C}}C_{\mathrm{AC}}{}^{\mathrm{A}}] = 8\pi T^{0i}, \quad (4.7)$$

$$\sigma_{ij}'+3\alpha'\sigma_{ij}+\sigma_{ik}\tau_{kj}-\tau_{kj}\sigma_{kj}+R_{ij}^{*}-\tfrac{1}{3}R^{*}\delta_{ij} = 8\pi[T_{ij}-\tfrac{1}{3}T_{kk}\delta_{ij}], \quad (4.8)$$

$$-6\alpha''-9(\alpha')^{2}-\tfrac{3}{2}\sigma_{ij}\sigma_{ij}-\tfrac{1}{2}R^{*} = 8\pi T_{kk}. \quad (4.9)$$

As in Misner (1968) the procedure adopted will be to use equations (4.6) and (4.8) to determine α and β as functions of t. Because of the Bianchi identities equation (4.9) will then be satisfied and equation (4.7) which relates the shear to the peculiar velocity of the matter, will hold at all values of t if it holds at one value. To analyse equation (4.8) for the β motions it is helpful to derive it from a Lagrangian. The Lagrangian of General Relativity is $\tfrac{1}{2}R+4\pi L_{m}$ where L_{m} is the Lagrangian of the matter. There are, however, two obstacles to using this Lagrangian to derive the equations for the models under consideration. First, the form of the metric does not allow g^{00} and g^{0i} to be varied. Thus the T^{00} and T^{0i} equations cannot be derived directly from the Lagrangian though, in fact, their time derivatives may be obtained as conservation laws because of the invariance of the Lagrangian under time translation and under the group of transformations of the $E^{\mathrm{A}}{}_{\mu}$ which leave the values of the structure constants unchanged.

* The cosmological constant is taken to be zero and units are such that $G = c = 1$.

The second difficulty arises from the fact that α and β are functions of t only. Thus, although the region in which the metric is varied can be chosen to be bounded in time, it may be unbounded in space. Such a variation would give the correct equations only if the divergence obtained in the variation of R were to vanish identically. This will be the case for models in class A. In general it does not seem possible to derive the equations for class B ($C_{AC}{}^{A}$ non-zero) models from a Lagrangian. However, for type V, a Lagrangian is not really needed since $R_{ij}^* - \frac{1}{3}R^*\delta_{ij}$ is zero and so equation (4.8) is relatively simple to integrate.

For class A models the part of the action which determines the motions is

$$\int L_\beta dt$$

where

$$L_\beta(\beta, \beta', t) = e^{3\alpha}(\tfrac{1}{2}\sigma_{ij}\sigma_{ij} + \tfrac{1}{2}R^* + 4\pi L_m). \quad (4.10)$$

The explicit time dependence occurs through α and through the matter Lagrangian. The normal Lagrangian for a perfect fluid is $2\rho(1+\epsilon)$ where ρ is the conserved density, ϵ is the internal energy and $\rho(1+\epsilon)$ is the total energy density μ. In these models conservation of ρ may be expressed as

$$\rho U^0 e^{3\alpha} = \text{constant}. \quad (4.11)$$

The variation in ρ induced by a variation in the metric is obtained from equation (4.11) where U^0 is regarded as being given by

$$U^0 = (1 - q^A q^B e^{2\alpha}(e^{2\beta})_{AB})^{-1/2}$$

and $q^A(t) = (U^0)^{-1}U^A$ is held fixed. However, from equations (3.7) and (3.9) the β motions may be derived more simply by replacing L_m by

$$-2a_b e^{-3\alpha}V_b - 4a_p e^{-4\alpha}V_p.$$

Then subtracting out a term independent of β, L_β may be redefined as

$$L_\beta = \tfrac{1}{2}e^{3\alpha}\sigma_{ij}\sigma_{ij} - e^{\alpha}V_g - 8\pi a_b V_b - 16\pi a_p e^{-\alpha}V_p \quad (4.12)$$

where $V_g = -\frac{1}{2}e^{2\alpha}(R^* - R^*|_{\beta=0})$. The Hamiltonian is

$$H_\beta = \tfrac{1}{2}e^{3\alpha}\sigma_{ij}\sigma_{ij} + e^{\alpha}V_g + 8\pi a_b V_b + 16\pi a_p e^{-\alpha}V_p. \quad (4.13)$$

It obeys

$$\frac{dH_\beta}{dt} = -\frac{\partial L_\beta}{\partial t} = -\frac{\partial L_\beta}{\partial \alpha}\alpha'. \quad (4.14)$$

The quantity $H_\beta e^{-3\alpha}$ may be thought of as the density of anisotropy energy since it occurs in the expansion equation in a similar way to the matter energy:

$$3(\alpha')^2 = 8\pi a_b e^{-3\alpha} + 8\pi a_p e^{-4\alpha}[\tfrac{4}{3} - \tfrac{2}{3}V_p - \tfrac{1}{3}(1+V_p)^{-2}] + H_\beta e^{-3\alpha} - \tfrac{1}{2}R^*|_{\beta=0}. \quad (4.15)$$

5. *Type IX*

5.1 *Dynamics*. Back to the time when the background radiation was emitted or last scattered the universe would have been matter dominated and the departures from isotropy would have been small. Thus the dynamical effect of photons and neutrinos will be neglected and β and U^i will be assumed small.

The β motions are derived from the Lagrangian

$$L_\beta = \tfrac{1}{2}e^{3\alpha}\sigma_{ij}\sigma_{ij} - e^{\alpha}V_g - 8\pi a_b V_b. \quad (5.1)$$

This may be considered as representing the motion of a particle with time dependent mass in a time dependent potential $V = e^{-2\alpha}V_g + 8\pi a_b e^{-3\alpha}V_b$. By equation (4.4),

$$V_g = \tfrac{1}{4}tr(e^{4\beta} - 2e^{-2\beta} + 1), \tag{5.2}$$

(Misner (1967)). For small β this gives a harmonic oscillator potential;

$$V_g = \beta_{ij}\beta_{ij} + o(\beta^3). \tag{5.3}$$

For small U^i and β,

$$V_b = -e^{-2\alpha}U_A U_B \beta_{AB}. \tag{5.4}$$

The effect of this is to shift the minimum of V from $\beta = 0$ to

$$\beta_{AB} = 4\pi e^{-3\alpha}[U_A U_B - \tfrac{1}{3}\delta_{AB}U_C U_C].$$

The motion in the potential well will, in general, possess angular momentum about the centre. The T^{0i} equations may be regarded as expressing conservation of this angular momentum. For small β, they are,

$$\epsilon_{ACD}[\beta_{AB}\beta_{BC}' - \beta_{AB}'\beta_{BC}] = 8\pi a_b e^{-3\alpha}U_D. \tag{5.5}$$

The matrix β may be expressed in terms of an orthogonal matrix O and a diagonal trace-free matrix D; $\beta = ODO^T$. Then $\beta' = O[\Omega D + D' - D\Omega]O^T$ where $\Omega = O^T O'$ represents the angular velocity of the principal axes of β. Thus

$$(2D\Omega D - D^2\Omega - \Omega D^2)_{AB} = 4\pi a_b e^{-3\alpha}\epsilon_{ABC}O_{DC}U_D, \tag{5.6}$$

showing that the principal axes rotate about the direction $O_{DC}U_D$. Solving for Ω,

$$\Omega_{12} = -4\pi a_b{}^{-3\alpha}[D_{11} - D_{22}]^{-2}O_{D3}U_D \text{ etc.} \tag{5.7}$$

The kinetic energy of the motion in the potential well is $\tfrac{1}{2}\sigma_{ij}\sigma_{ij}$. For small β this is

$$\tfrac{1}{2}\,tr[2\Omega D\Omega D - 2\Omega^2 D^2 + (D')^2] = \tfrac{1}{2}\,tr[(D')^2] + 16\,\pi^2 a_b{}^2 e^{-6\alpha}[(D_{11} - D_{22})^{-2}(O_{D3}U_D)^2 + \text{etc.}]. \tag{5.8}$$

The total energy is given by

$$e^{-3\alpha}H_\beta = \tfrac{1}{2}\beta_{AB}'\beta_{AB}' + e^{-2\alpha}\beta_{AB}\beta_{AB}, \tag{5.9}$$

where V_b has been neglected since by equation (5.5), U_A is of order $\beta\beta'$. For a given value of U_A, the minimum total energy is $4\sqrt{2\pi a_b}e^{-4\alpha}\,(U_A U_A)^{1/2}$ which would be achieved by a ' circular ' orbit, that is a motion in which the eigenvalues of β remained constant while one principal direction lay along U_A and the other two rotated about it.

If α were constant, the system would execute simple harmonic motion with period $\sqrt{2\pi e^\alpha}$ in the five dimensional β space. However, α is, of course, varying with time and the average Hubble time $(\alpha')^{-1}$ is less than the period of oscillation except when α is greater than $\alpha_0 - \log(1 + 2\pi^{-2})$ where α_0 is the maximum value of α.

For $\alpha \ll \alpha_0$,

$$e^{3\alpha} = \tfrac{9}{16}\,e^{\alpha_0}t^2. \tag{5.10}$$

Thus for small β,

$$\frac{d^2\beta}{dt^2}+\frac{2}{t}\frac{d\beta}{dt}+2\left[\frac{9}{16}e^{\alpha_0}t^2\right]^{-2/3}\beta = 0, \tag{5.11}$$

$$\beta = Y^{-3/2}\,[AJ_{3/2}(Y)+BJ_{-3/2}(Y)], \tag{5.12}$$

where A and B are constant matrixes and $Y = 3\sqrt{2}\,[\frac{9}{16}\,e^{\alpha_0}]^{-1/3}t^{1/3}$. Thus

$$\beta = F[1-\tfrac{4}{5}\,e^{\alpha-\alpha_0}+\ \dots\ ]+Ge^{-3/2(\alpha-\alpha_0)}[1+e^{\alpha-\alpha_0}+\ \dots\ ] \tag{5.13}$$

where F and G are constant matrices.

5.2 *Observations*. If the model were isotropic (i.e. $\beta = 0$) the equation of a null geodesic would be, K_{A} constant, $K^0 = e^{-\alpha}\,[K_{\mathrm{A}}K_{\mathrm{A}}]^{1/2}$. Thus by Section 3.2, the observed temperature of the radiation will be to the first order,

$$T_R = T_{\mathrm{E}}\lambda^{-1}[1+p^iU_R{}^i(1-\lambda)-p^ip^j(\beta_R-\beta_{\mathrm{E}})_{ij}], \tag{5.14}$$

where p^i is a unit space-like vector in the direction of observation and

$$\lambda = \exp\,(\alpha_R-\alpha_{\mathrm{E}}).$$

If the radiation is left over from the primeval fireball the value of λ will be between 8 and 1000 depending on the thermal history of the intergalactic gas.

The observations of Partridge & Wilkinson (1967) near the equator give values of $0{\cdot}16\pm0{\cdot}07$ per cent for the 12 h component and $0{\cdot}03\pm0{\cdot}07$ per cent for the 24 h component. Taking these values to be typical of any circle of observation, one sees that $U_R{}^i(\lambda-1)$ must be less than 10^{-3} showing that the assumption of small U^i was justified. By equation (3.10), the present value of the vorticity must be less than $10^{-3}\,(\lambda-1)\exp\,(-\alpha_R)$. Since $\exp\,\alpha_R$ must be of the order of the Hubble radius $1{\cdot}3\times10^{10}$ light years, or greater, it follows that the vorticity is less than $7\times10^{-14}\,(\lambda-1)^{-1}$ rad yr^{-1}.

The components of $\beta_R-\beta_{\mathrm{E}}$ must be less than about 2×10^{-3}. Knowing this it is not possible to place an upper limit on the present value of β since F and G in equation (5.14) could be correlated so that $\beta_R-\beta_{\mathrm{E}}$ vanished at the present though not at other times. However, it seems more reasonable to believe that F and G are unrelated. Then F would be less than about

$$2{\cdot}5\times10^{-3}\,(1-\lambda^{-1})^{-1}\exp\,(\alpha_0-\alpha_R),$$

and G less than about $2\times10^{-3}\,(\lambda^{3/2}-1)$. Provided that $\exp\,(\alpha_0-\alpha_R)$, the ratio of the maximum radius of the universe to the present radius, is less than 100, it can be seen that the assumption of small β was justified.

6. *Type V*

6.1 *Dynamics*. In models of type V there is a distinguished two-plane in the surfaces of homogeneity defined by the covector $E^{\mathrm{B}}{}_{\mu}C_{\mathrm{AB}}{}^{\mathrm{A}}$. The invariant vector fields $E_2{}^{\mu}$ and $E_3{}^{\mu}$ lie in this plane and may be chosen so that at the present time they have equal magnitude and are mutually orthogonal. The field $E_1{}^{\mu}$ may be chosen to have the same magnitude and to be orthogonal to the plane at the present time. This makes β_R zero. As before the effect of radiation will be neglected and U^i assumed small. With these approximations

$$\mu_b = a_b e^{-3\alpha}, \qquad (6.1)$$

$$3(\alpha')^2 = 8\pi a_b e^{-3\alpha} + 3e^{-2\alpha}, \qquad (6.2)$$

$$3\sigma_{1i} = 8\pi a_b e^{-3\alpha} U_i, \qquad (6.3)$$

$$\sigma_{ij}' + 3\alpha' \sigma_{ij} = 0. \qquad (6.4)$$

From equation (6.4), $\sigma_{ij} = A_{ij} e^{-3\alpha}$ where A_{ij} is a constant matrix and A_{1i} are given by equation (6.3). By rotating $E_2{}^{\mu}$ and $E_3{}^{\mu}$ in the distinguished two plane A_{23} can be made zero. For a low density model, that is one in which μ_b is much less than $(\alpha')^2$, the square of the reciprocal of the average Hubble time, $e^{\alpha} = t$. Thus

$$\beta_{ij} = \tfrac{1}{2} A_{ij}[t_R{}^{-2} - t^{-2}]. \qquad (6.5)$$

6.2 *Observations.* These are more complicated since even in an isotropic model with β identically zero, the components of the tangent vector to a null geodesic are not constant. Expressing K_A as $K_1 = K \cos\theta$, $K_2 = K \sin\theta \cos\phi$ and $K_3 = K \sin\theta \sin\phi$, one has from equation (3.2) that, for β zero, K and ϕ are constant and θ obeys

$$\theta' = -e^{-\alpha} \sin\theta.$$

Thus for $\mu_b \ll (\alpha')^2$

$$\theta = 2\cot^{-1} ft, \qquad (6.6)$$

where f is a constant. Using this the observed temperature may be calculated according to Section 3.2: the isotropic component is $T_E \lambda^{-1}$. The term $U_R{}^i K_R{}^i$ gives rise to a dipole component

$$T_E \lambda^{-1}[U_R{}^1 \cos\theta + U_R{}^2 \sin\theta \cos\phi + U_R{}^3 \sin\theta \sin\phi].$$

The term $U_E{}^i K_E{}^i$ gives a component,

$$-T_E(1 + a^2\lambda^{-2})^{-1}[U_R{}^1(a^2\lambda^{-2} - 1) + 2a\lambda^{-1}(U_R{}^2 \cos\phi + U_R{}^3 \sin\phi)]$$

where $a = \cot\theta/2$. In a low density model λ would probably be about 10^3 if the radiation was left over from the primeval fireball. Thus for θ greater than 5 minutes of arc, the anisotropic part of this component would be

$$-2T_E \lambda^{-1} \cot\frac{\theta}{2} [U_R{}^2 \cos\phi + U_R{}^3 \sin\phi].$$

The anisotropic part of the component arising from $[K_E{}^i K_E{}^i]^{1/2}$ would have the approximate form for θ greater than 5 minutes of arc,

$$4T_E \lambda^{-1} \exp(-2\alpha_R)\left[\cot^2\frac{\theta}{2} \log \lambda (A_{11} - A_{22}\cos^2\phi - A_{33}\sin^2\phi) + \cot\frac{\theta}{2}(\lambda - 1)(A_{12}\cos\phi + A_{13}\sin\phi)\right].$$

This pattern of variation is very different from that in a spatially closed model and would provide a possible new cosmological test if observations could be made over the whole sky. Near the direction $\theta = 0$ there could be very marked departures from the mean temperature and the $\cos\phi$ and $\cos 2\phi$ behaviour of these departures should make it possible for them to be distinguished from the effects of inhomogeneities. No conclusion on this question can be drawn from the measurements of Partridge & Wilkinson (1967) since their observations were confined to one circle and the chances would be that this would not pass very near the direction $\theta = 0$. Assuming that the circle has some 'typical' orientation, one can place limits of about $2\cdot 5\times 10^{-36}$/(present density in g cm^{-3}) on $U_R{}^2$ and $U_R{}^3$ of about 10^{-3} on $U_R{}^1$ and of about 10^{-4} exp $(2\alpha_R)$ on A_{22} and A_{33}, showing that the assumption of small U^i and σ was justified. With a present density of 10^{-31} g cm^{-3} the neglect of μ_b in equation (6.2) would not be justified all the way back to the time when the radiation decoupled. However, the effect of including this term would be to increase slightly the observed temperature anisotropy and so to give even lower limits on $U_R{}^i$ and A_{ij}.

The vorticity depends only on U^2 and U^3. Thus the limit on the present value is 2×10^{-46} (present density)$^{-1}$ rad yr^{-1}.

7. *Viscosity*. The limits obtained on the vorticity and on the peculiar velocity of the matter imply that they are small back to the time when the radiation dominates over the matter. At earlier times it follows from equation (3.8) that the peculiar velocity is practically independent of α and so should remain small unless the universe becomes very anisotropic. The question then arises as to whether this effectively zero value represents the initial condition or whether the peculiar velocity was once large but has been reduced by dissipative processes. To investigate this possibility, the effect of viscosity on the peculiar velocity will be considered.

The anisotropic stresses from viscosity can be represented as $-2\eta\sigma_{(m)\mu\nu}$ where η is the coefficient of viscosity and

$$\sigma_{(m)\mu\nu} = U_{(\lambda;\rho)}h^\lambda{}_\mu h_\nu{}^\rho - \tfrac{1}{3}h_{\mu\nu}U^\lambda{}_{;\lambda} \qquad (7.1)$$

is the shear tensor of the matter. They cause acceleration of the flow-lines:

$$\tfrac{4}{3}\mu_\rho U^\mu{}_{;\nu}U^\nu = -\tfrac{1}{3}\mu_{\rho;\nu}h^{\mu\nu} + 2\eta\sigma_{(m)}{}^{\lambda\rho}{}_{;\rho}h_\lambda{}^\mu. \qquad (7.2)$$

To first order in β_{ij} and U^i, this gives,

$$(e^{-\alpha}U_{\mathrm{A}})' = -\eta[16\pi e^{-\alpha}U_{\mathrm{A}} + \tfrac{3}{4}a_p{}^{-1}e^\alpha U_{\mathrm{E}}(C_{\mathrm{CA}}{}^{\mathrm{B}}C_{\mathrm{BE}}{}^{\mathrm{C}} + C_{\mathrm{CA}}{}^{\mathrm{B}}C_{\mathrm{CE}}{}^{\mathrm{B}} - C_{\mathrm{BC}}{}^{\mathrm{B}}C_{\mathrm{CE}}{}^{\mathrm{A}} - C_{\mathrm{BC}}{}^{\mathrm{B}}C_{\mathrm{AE}}{}^{\mathrm{C}})]. \qquad (7.3)$$

In the early stages the first term on the right gives the dominant contribution which tends to reduce the peculiar velocity and the vorticity.

In Misner (1968) it was shown that there would have been a large viscosity produced by collisions between neutrinos and electron–positron pairs when the temperature was about 2×10^{10} °K. However, this viscosity would last only for a time of about $(64\,\pi\eta)^{-1}$. Thus the peculiar velocity would be reduced by a factor of $e^{0\cdot 25}$ only. There would also be a viscosity during the decoupling of the matter from the photons. Again however, the viscosity would only act for a short time so

$$16\pi\int \eta\, dt$$

would be small. It appears, therefore, that viscosity is not very effective in reducing a peculiar velocity which is small compared to the speed of light. If it is large, the damping is much more rapid and it seems possible that neutrino viscosity could reduce $(U^iU^i)^{1/2}$ from an arbitrarily large initial value to order unity. It would then remain of order unity until the matter decoupled from the radiation and this would be compatible with observation only if the universe were open and the present density were less than the very low value of

$$2 \cdot 5 \times 10^{-33} \text{ g cm}^{-3}.$$

Otherwise, one would have to appeal to Machian initial conditions to explain the observations.

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